I.　デルタ関数とヘヴィサイド関数に関係する式

A.　定義

1.　ヘヴィサイド関数

$$\theta(x) = \begin{cases} 1, & (x > 0) \\ 1/2, & (x = 0) \\ 0, & (x < 0) \end{cases} \tag{1}$$

2.　符号関数

$$\mathrm{sgn}(x) = \begin{cases} 1, & (x > 0) \\ 0, & (x = 0) \\ -1, & (x < 0) \end{cases} \tag{2}$$

3.　矩形関数

$$\mathrm{rect}(x) = \begin{cases} 0, & (|x| > 1/2) \\ 1/2, & (x = 1/2) \\ 1, & (|x| < 1/2) \end{cases} \tag{3}$$

4.　デルタ関数

デルタ関数 $\delta(x)$ は、

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0) \tag{4}$$

を満たす関数である。これを満たす $\delta(x)$ は、

$$\delta(x) = \begin{cases} 0, & (x \neq 0) \\ \infty, & (x = 0) \end{cases} \tag{5}$$

の特徴を持つ。

5. フーリエ変換

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{6a}$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} \frac{dk}{2\pi} \tag{6b}$$

B. 性質

$$\theta(-x) = -\theta(x) + 1 \tag{7}$$

$$\theta(ax) = \theta(x), \quad (a > 0) \tag{8}$$

$$\mathrm{sgn}(x) = 2\theta(x) - 1 \tag{9}$$

$$\mathrm{rect}\left(\frac{x-b}{a}\right) = \theta\left(x - b + \frac{a}{2}\right) - \theta\left(x - b - \frac{a}{2}\right) \tag{10}$$

$$\theta(t) = i \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i\varepsilon} \tag{11a}$$

$$\theta(t) = i \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{-\omega + i\varepsilon} \tag{11b}$$

$$\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \tag{11c}$$

$$\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tag{11d}$$

C. 積分

$$\int_A^B f(x)\delta(x-a)dx = \left[\theta(a-A) - \theta(a-B)\right] f(a) \tag{12a}$$

$$= \left[\theta(B-a) - \theta(A-a)\right] f(a) \tag{12b}$$

$$\int_A^B f(x)\theta(x-a)dx = \theta(A-a)\int_A^a f(x)dx + \theta(B-a)\int_a^B f(x)dx \tag{13a}$$

$$\int_A^B f(x)\theta(a-x)dx \;=\; \theta(a-A)\int_A^a f(x)dx + \theta(a-B)\int_a^B f(x)dx \tag{13b}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(k)}{k} e^{ikx} \frac{dk}{2\pi} \;&=\; \frac{1}{4}\Big[\mathrm{sgn}(x+1) - \mathrm{sgn}(x-1)\Big] && \text{(14a)}\\
&=\; \frac{1}{2}\Big[\theta(x+1) - \theta(x-1)\Big] && \text{(14b)}\\
&=\; \frac{1}{2}\ \mathrm{rect}\left(\frac{x}{2}\right) && \text{(14c)}
\end{aligned}$$

D. 畳み込み積分

$$V(x)f(x) = \int \left[\int V(k-k')f(k')\frac{dk'}{2\pi}\right] e^{ikx}\frac{dk}{2\pi} \tag{15}$$

$$\delta(x)f(x) = \int \left[\int f(k')\frac{dk'}{2\pi}\right] e^{ikx}\frac{dk}{2\pi} \tag{16}$$

E. グリーン関数

$$\left[\frac{\partial^2}{\partial t^2} - a\frac{\partial^2}{\partial x^2}\right] G(x,t;x',t') = \delta(x-x')\delta(t-t') \tag{17}$$

$$\begin{aligned}
G(x,t;x',t') \;&=\; \frac{\theta(t-t')}{2\sqrt{a}}\left[\theta\Big((x-x') - \sqrt{a}(t-t')\Big) - \theta\Big((x-x') + \sqrt{a}(t-t')\Big)\right] && \text{(18a)}\\
&=\; \frac{\theta(t-t')}{2\sqrt{a}}\ \mathrm{rect}\left(\frac{x-x'}{2\sqrt{a}(t-t')}\right) && \text{(18b)}
\end{aligned}$$